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ON THE VECTOR SPACE OF 0-CONFIGURATIONS

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Let α be a rational-valued set-function on the n-element set X i.e. $\alpha(B)\in Q$ for every $B\subseteqq X$. We say that α defines a 0-configuration with respect to $\mathcal{A}\subseteqq 2^X$ if for every $A\in\mathcal{A}$ we have $\sum_{A\subseteqq B\subseteqq X}\alpha(B)=0$. The 0-configurations form a vector space of dimension $2^n-|\mathcal{A}|$ (Theorem 1). Let $0\leqq t<k\leqq n$ and let $\mathcal{A}=\{A\subseteqq X: |A|\leqq t\}$. We show that in this case the 0-configurations satisfying $\alpha(B)=0$ for $|B|>k$ form a vector space of dimension $\sum_{t<i\leqq k}\binom{n}{i}$, we exhibit a basis for this space (Theorem 4). Also a result of Frankl, Wilson [3] is strengthened (Theorem 6).

1. Introduction and statement of the results

Let $X=\{x_1, \dots, x_n\}$ be a finite set of n elements. For an element A of $2^X=\{F\colon F\subseteqq X\}$ we define the monomial $p(A)=\prod_{x\in A} x, p(\emptyset)=1$.

Set $V=V(2^X)=\left\{\sum_{A\subseteqq X}\alpha(A)p(A)\colon \alpha(A) \text{ is rational}\right\}$, i.e. V is the set of all square-free polynomials in the variables $x_1, \dots, x_n$. Of course, V is a vector space of dimension 2^n over Q, the field of rationals.

For a family of subsets, $\mathcal{A}\subseteqq 2^X$ we define
$V(\mathcal{A})=\left\{\sum_{A\subseteqq X}\alpha(A)p(A)\colon\alpha(A) \text{ is rational, } \alpha(A)=0 \text{ unless } A\in\mathcal{A}\right\}$.

For an $\mathcal{A}\subseteqq 2^X$ and $f=\sum_{A\subseteqq X}\alpha(A)p(A)\in V$ we say that f is *$\mathcal{A}$-orthogonal* or a *0-configuration with respect to $\mathcal{A}$* if

$$\sum_{A\subseteqq B\subseteqq X}\alpha(B)=0 \quad \text{holds for all } A\in\mathcal{A}. \tag{1}$$

We prove:

Theorem 1. *The set $V^*(\mathcal{A})$ of all $\mathcal{A}$-orthogonal elements of $V(2^X)$ is a vector space of dimension $2^n-|\mathcal{A}|$, moreover $V(\mathcal{A})\cap V^*(\mathcal{A})=\{0\}$.*

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In [1] the following more general definition was considered:

Definition 2. *For* $\mathcal{A}, \mathcal{B} \subseteq 2^X$ *we set*

$$V^*(\mathcal{A}, \mathcal{B}) = \{f \in V^*(\mathcal{A}) \colon f \in V(\mathcal{B})\} = V^*(\mathcal{A}) \cap V(\mathcal{B}).$$

The special case $\mathcal{A} = \binom{X}{\leqq t} = \{A \in 2^X \colon |A| \leqq t\}$, $\mathcal{B} = \binom{X}{k} = \{B \in 2^X \colon |B| = k\}$ ($k > t$, non-negative integers) was considered by Graver, Jurkat [5], and Graham, Li, Li [4]. The elements of $V^*(\mathcal{A}, \mathcal{B})$ are called in this special case 0-designs.

Remark. Note that for $B \subseteq \binom{X}{k}$ we have

$$V^*\left(\binom{X}{\leqq t}, \mathcal{B}\right) = V^*\left(\binom{X}{t}, \mathcal{B}\right),$$

and for $\mathcal{A} \subseteq \mathcal{A}' \subseteq 2^X$, $\mathcal{B}' \subseteq \mathcal{B} \subseteq 2^X$

$$V^*(\mathcal{A}, \mathcal{B}) \supseteq V^*(\mathcal{A}', \mathcal{B}') \quad \text{holds.}$$

Theorem 3. ([5], [4]). *The space* $V^*\left(\binom{X}{t}, \binom{X}{k}\right)$ *is generated by the polynomials* $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \dots (x_{i_{2t+1}} - x_{i_{2t+2}}) x_{i_{2t+3}} \dots x_{i_{k+t+1}}$. ($x_{i_1}, \dots, x_{i_{k+t+1}}$ are distinct elements of X), $\dim V^*\left(\binom{X}{t}, \binom{X}{k}\right) = \binom{n}{k} - \binom{n}{t}$.

Here we prove

Theorem 4. *A basis of the space* $V^*\left(\binom{X}{\leqq t}, \binom{X}{\leqq k}\right)$ *is formed by the polynomials* $(x_{i_1} - 1) \dots (x_{i_{t+1}} - 1)\ x_{i_{t+2}} \dots x_{i_l}$ *where* $t+2 \leqq l \leqq k$ *and* $1 \leqq i_1 < \dots < i_l \leqq n$. *Thus* $\dim \left(V^*\left(\binom{X}{\leqq t}, \binom{X}{\leqq k}\right)\right) = \sum_{t+1 \leqq l \leqq k} \binom{n}{l}$.

Remark. In [2] the case $t=2$, $k=n$ was considered. There the terminology 0-measure (or isometry) was used and a entirely different generator system was exhibited.

The following theorem of Ray—Chaudhuri, Wilson [6] can be formulated in terms of 0-designs.

Theorem 5. [6]. *Suppose* $\mathcal{B} \subset \binom{X}{k}$ *is such that* $|B \cap B'|$ *takes at most t values for* $B \neq B' \in \mathcal{B}$. *Then* $V^*\left(\binom{X}{t}, \mathcal{B}\right) = \{0\}$ *i.e.* $V(\mathcal{B})$ *contains no non-trivial* 0-*design, and consequently*

$$|\mathcal{B}| = \dim V(\mathcal{B}) \leqq \dim V\left(\binom{X}{k}\right) - \dim V^*\left(\binom{X}{t}, \binom{X}{k}\right) = \binom{n}{t}.$$

Singhi [7] pointed out that one can localize Theorem 5, i.e. in any 0-design $f=\sum_{B\in\binom{X}{k}}\alpha(B)p(B)$ one can find a $B\in\binom{X}{k}$ such that $\alpha(B)\neq 0$ and $|\{|B\cap B'|: B\neq B'\in\binom{X}{k}, \alpha(B')\neq 0\}|\geqq t+1$.

We prove a similar strengthening of a theorem of Frankl—Wilson [3].

Theorem 6. *Suppose* $f=\sum_{A\subseteq X}\alpha(A)p(A)\in V^*\left(\binom{X}{\leqq t},\binom{X}{\leqq k}\right)$. *Then there exists an* $A\subseteq X$ *such that* $\alpha(A)\neq 0$ *and*

$$|\{|A\cap A'|:\ A\subsetneqq A'\subseteqq X,\ \alpha\{A')\neq 0\}|\geqq t+1.$$

2. The proof of the results

Proof of Theorem 1. The fact that $V^*(\mathscr{A})$ is a vector space is evident. The solutions of (1) for a fixed $A\in 2^X$ form a subspace of dimension 2^n-1 in V — as the solution of any non-trivial homogenous linear equation. Now obviously $V^*(\mathscr{A})=\bigcap_{A\in\mathscr{A}}V^*(\{A\})$. Thus $V^*(\mathscr{A})$ is the intersection of $|\mathscr{A}|$ subspaces of dimension 2^n-1, yielding

$$\dim V^*(\mathscr{A})\geqq 2^n-|\mathscr{A}|. \tag{2}$$

As $\dim V(\mathscr{A})=|\mathscr{A}|$, $\dim V^*(\mathscr{A})=2^n-|\mathscr{A}|$ will follow from (2) if we establish $V(\mathscr{A})\cap V^*(\mathscr{A})=\{0\}$. To prove this let $f=\sum_{A\in\mathscr{A}}\alpha(A)p(A)$ be an arbitary non-zero element of $V(\mathscr{A})$. Choose an $A\in\mathscr{A}$ such that $\alpha(A)\neq 0$ but $\alpha(A')=0$ for every $A'\supset A$. Then checking for A the condition (1) we conclude $f\notin V^*(\mathscr{A})$. ∎

Proof of Theorem 4. In the case $n\leqq t$ obviously $V^*\left(\binom{X}{\leqq t},\binom{X}{\leqq k}\right)=0$, thus the statement is true. We apply induction on n, simultaneously for all k, t. Let $f=\sum_{A\subseteq X}\alpha(A)p(A)$belong to $V^*\left(\binom{X}{\leqq t},\binom{X}{\leqq k}\right)$. Then we can write $f=f_0+f_1$ where

$$f_0=\sum_{x_n\notin A\subseteq(X-\{x_n\})}\alpha(A)p(A),\quad f_1=\sum_{x_n\notin A\subseteq(X-\{x_n\})}\alpha(A\cup\{x_n\})p(A)x_n.$$

Let us set $f_2=f_1/x_n$. Then

$$(f_0+f_2)\in V^*\left(\binom{X-\{x_n\}}{\leqq t},\binom{X-\{x_n\}}{\leqq k}\right),\quad\text{and}$$

$$f_2\in V^*\left(\binom{X-\{x_n\}}{\leqq t-1},\binom{X-\{x_n\}}{\leqq k-1}\right).$$

As $f=(f_0+f_2)+(x_n-1)f_2$ the induction hypothesis yields the decomposition of f into a linear combinations of polynomials in V, each of them of the form $(x_{i_1}-1)\dots(x_{i_{t+1}}-1)x_{i_{t+2}}\dots x_l$, $l\leqq k$.

For f_0+f_2 there are no problems, however for $(x_n-1)f_2$ the monotonicity is violated for every term g in the decomposition of f_2 having $l>t+1$.

For such g we can write

$$(x_n-1)g = \sum_{j=t+1}^{l}\left(g\frac{x_{i_g}-1}{\prod\limits_{t+1\leqq v\leqq j} x_{i_v}}x_n - g\frac{x_{i_g}-1}{\prod\limits_{t+1\leqq v\leqq j} x_{i_v}}\right)+(x_{i_1}-1)\dots(x_{i_t}-1)(x_n-1),$$

which procures a decomposition with the desired property.

Now we calculate the dimension of the space $W=V^*\left(\binom{X}{\leqq t},\binom{X}{\leqq k}\right)=$ $=V^*\left(\binom{X}{\leqq t}\cup\binom{X}{\geqq k+1}\right)$, thus Theorem 1 yields $\dim W=\sum_{t+1\leqq l\leqq k}\binom{n}{l}$, proving the second part of the theorem.

Since in new generator systems the number of polynomials is just $\dim W$, they are linearly independent, i.e. they form a basis. ∎

Remark. The proof also shows that these polynomials form a system of generators for $V^*\left(\binom{X}{\leqq t},\binom{X}{\leqq k}\right)$ as a **Z**-module, i.e. if f has integer coefficients then it can be obtained as an integer linear combination of the generators.

Proof of Theorem 6. For $B\subseteqq 2^X$ we define matrices M_i, $0\leqq i\leqq t$. For that let $A_1, \dots, A_{\binom{n}{i}}$ be a fixed ordering of the elements of $\binom{X}{i}$ and $B_1, \dots, B_m$ of those of $\mathscr{B}$. i.e. $|\mathscr{B}|=m$. Now for $1\leqq r\leqq m$, $1\leqq s\leqq\binom{n}{i}$ the element

$$m_i(r, s) = \begin{Bmatrix} 0 & \text{if} & A_s \nsubseteq B_r \\ 1 & \text{if} & A_s \subseteqq B_r \end{Bmatrix}.$$

Let M be the m by $\binom{n}{t}+\binom{n}{t-1}+\dots+\binom{n}{0}$ matrix which we obtain by putting side by side $M_t, \dots, M_0$. Let us denote by u_i the i'th row vector of M. In this case $f=\sum_{B_i\in\mathscr{B}}\alpha(B_i)p(B_i)\in V^*\left(\binom{X}{\leqq t}\right)$ is equivalent to $\sum_{i=1}^{m}\alpha(B_i)u_i=0$ i.e. in that case the row vectors of M are not independent and consequently the rank of M is less than m.

Suppose now that $\mathscr{B}\subseteqq 2^X$ is such that for every $B\in\mathscr{B}$, $|B\cap B'|$ takes at most t values different from $|B|$. In view of the above observations it is sufficient to show that in this case the rank of M is at least m.

To do this let $v_{i,j}$ denote the i'th column vector of M_j, $1\leqq i\leqq m$, $0\leqq j\leqq t$, and let W be the vector space spanned by these vectors over Q.

Let us calculate the matrices $N_j=M_jM_j^T$ for $0\leqq j\leqq t$. N_j is an m by m matrix with column vectors w_i^j, $1\leqq i\leqq m$, $w_i^j\in W$. The (r, s)-element of N_j is $\binom{|B_r\cap B_j|}{j}$, $1\leqq r, s\leqq m$.

Without loss of generality we assume $|B_r|\geqq|B_s|$ for $1\leqq r<s\leqq m$. Fix r, $1\leqq r\leqq m$ and let $l_1, \dots, l_p$ be the different values of $|B_r\cap B_s|$ for $r<s\leqq m$. Thus,

by our assumption $p \leqq t$, there exist rational constanst c_j such that $(x-l_1)\dots$ $\dots(x-l_p) = \sum_{1 \leqq j \leqq t} c_j \binom{x}{j}$. Let us set $u_r = \sum_{j=0}^{t} c_j w_r^j$ and let N be the m by m matrix formed by the column vectors $u_r \in W$, $1 \leqq r \leqq m$.

By definition N is an upper-triangular matrix with non-zero diagonal (the t'th diagonal entry is $\prod_{i=1}^{p}(|B_r|-l_i) \neq 0$, while the (r, s)-entry for $r<s$ is $\prod_{i=1}^{p}(|B_r \cap B_s|-l_i)=0$), thus N has full rank m. As the columns of N are from W, we deduce $\dim W = \text{rank } M \geqq m$. ∎

Open problem. *Find a basis for $V^*(\mathscr{A}, \mathscr{B})$ in the general case, in particular determine* dim $V^*(\mathscr{A}, \mathscr{B})$.

In the particular case $\mathscr{A} = \binom{X}{t}$, $\mathscr{B} = \binom{X}{\leqq k}$ a basis can be obtained from the basis in Theorem 4 by adding all the monomials of degree less than t.

It is not hard to see that in the case $\mathscr{A} \subset \mathscr{B}$ we have $\dim V^*(\mathscr{A}, \mathscr{B}) = |\mathscr{B}| - |\mathscr{A}|$.

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